The Generalized Fermat Equation

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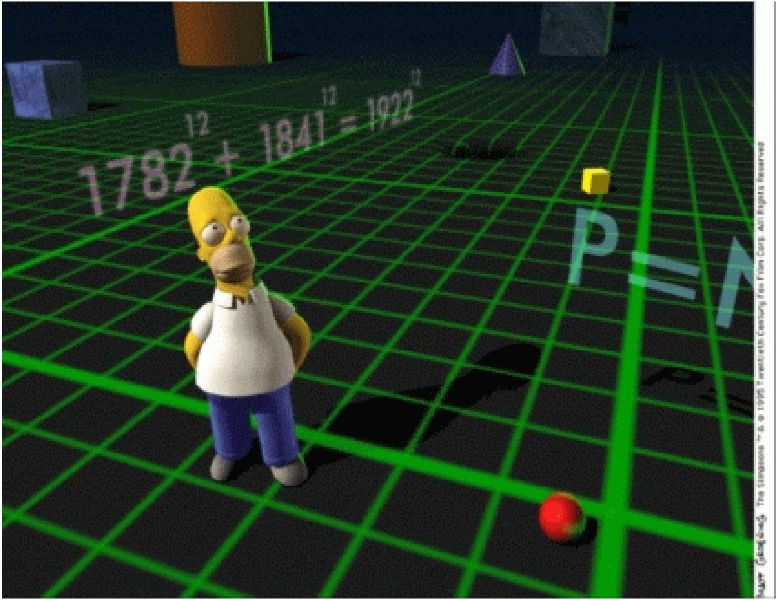
$1782^{12} + 1841^{12} = 1922^{12}$

Is it true that ...

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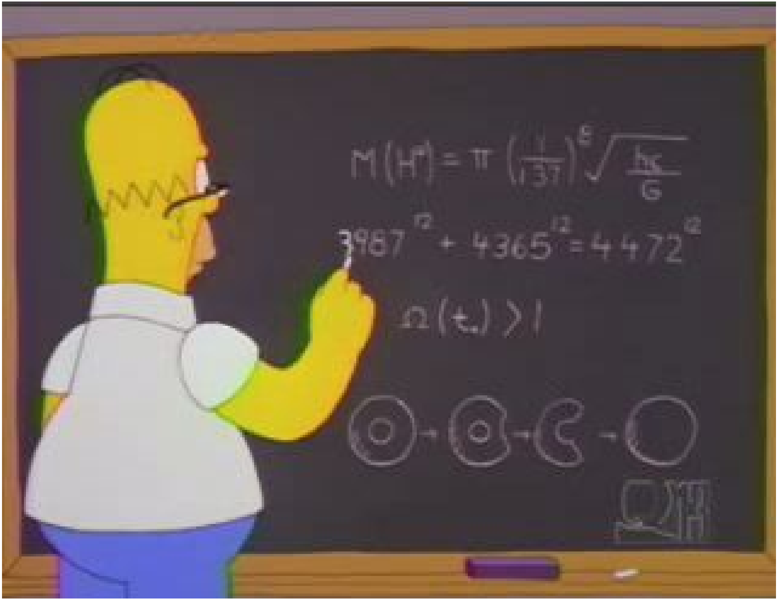
We have

$$\left(1782^{12} + 1841^{12}\right)^{1/12} = 1921.9999999\ldots.$$


$3987^{12} + 4365^{12} = 4472^{12}$

Perhaps

$$3987^{12} + 4365^{12} = 4472^{12}?$$

In this case

$$\left(3987^{12} + 4365^{12}\right)^{1/12} = 4472.00000000\ldots.$$

A Diophantine equation : Generalized Fermat

We consider the equation

$$x^p + y^q = z^r$$

where x, y and z are integers, and p, q and r are positive integers with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

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e.g.

$$1313620685576771842575055889 6^{11} + 2030141059527738302161450011 2^{11} = 3455720797824 4^{23}.$$

What's going on here?

$$13136206855767718425750558896^{11} + 20301410595277383021614500112^{11} = 34557207978244^{23}.$$

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then

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This is just the case $(a, b, n, t) = (11, 17, 11, 2)$.

A Diophantine equation : Generalized Fermat

We consider the equation

$$x^p + y^q = z^r$$

where x, y and z are relatively prime integers, and p, q and r are positive integers with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

- $(p, q, r) = (n, n, n)$: Fermat's equation
- $y = 1$: Catalan's equation
- considered by Beukers, Granville, Tijdeman, Zagier, Beal (and many others)

A simple case

$$x^p + y^q = z^r$$

where x, y and z are relatively prime integers, and p, q and r are positive integers with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

- $(p, q, r) = (2, 6, 3), (2, 4, 4), (4, 4, 2), (3, 3, 3), (2, 3, 6)$
- each case corresponds to an elliptic curve of rank 0
- the only coprime nonzero solutions is with $(p, q, r) = (2, 3, 6)$ – corresponding to $3^2 - 2^3 = 1$

For example : $x^3 + y^3 = z^3$

We write

$$Y = \frac{36(x-y)}{x+y} \quad \text{and} \quad X = \frac{12z}{x+y},$$

so that

$$Y^2 = X^3 - 432.$$

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so that

$$Y^2 = X^3 - 432.$$

This is 27A in Cremona's tables – it has rank zero and

$$E(\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z}.$$

A less simple case

$$x^p + y^q = z^r$$

where x, y and z are relatively prime integers, and p, q and r are positive integers with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

- $(2, 2, r), (2, q, 2), (2, 3, 3), (2, 3, 4), (2, 4, 3), (2, 3, 5)$
- in each case, the coprime integer solutions come in finitely many two parameter families (the canonical model is that of Pythagorean triples)
- in the $(2, 3, 5)$ case, there are precisely 27 such families (as proved by J. Edwards, 2004)

Back to

$$x^p + y^q = z^r$$

where x, y and z are relatively prime integers, and p, q and r are positive integers with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

Some solutions

$$1^n + 2^3 = 3^2,$$

$$2^5 + 7^2 = 3^4,$$

$$3^5 + 11^4 = 122^2,$$

$$2^7 + 17^3 = 71^2,$$

$$7^3 + 13^2 = 2^9,$$

$$43^8 + 96222^3 = 30042907^2,$$

$$33^8 + 1549034^2 = 15613^3,$$

$$17^7 + 76271^3 = 21063928^2,$$

$$1414^3 + 2213459^2 = 65^7,$$

$$9262^3 + 15312283^2 = 113^7.$$

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Every such solution has $\min\{p, q, r\} = 2$.

Conjecture (strong version $\geq$ \$1,000,000)

There are no additional solutions.

What we know

Theorem (Darmon and Granville) If A, B, C, p, q and r are fixed positive integers with

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

then the equation

$$Ax^p + By^q = Cz^r$$

has at most finitely many solutions in coprime nonzero integers x, y and z.

The state of the art (?)

(p, q, r)	reference(s)
(n, n, n)	Wiles, Taylor-Wiles
$(n, n, k), k \in \{2, 3\}$	Darmon-Merel, Poonen
$(2n, 2n, 5)$	B.
$(2, 4, n)$	Ellenberg, B-Ellenberg-Ng, Bruin
$(2, 6, n)$	B-Chen, Bruin
$(2, n, 4)$	B-Skinner, Bruin
$(2, n, 6)$	BCDY
$(3j, 3k, n),\ j, k \geq 2$	immediate from Kraus
$(3, 3, 2n)$	BCDY
$(3, 6, n)$	BCDY
$(2, 2n, k),\ k \in \{9, 10, 15\}$	BCDY
$(4, 2n, 3)$	BCDY

The state of the art : continued

(p, q, r)	reference(s)
$(3, 3, n)^*$	Chen-S, Kraus, Bruin, Dahmen
$(2, 2n, 3)^*$	Chen, Dahmen, S
$(2, 2n, 5)^*$	Chen
$(2m, 2n, 3)^*$	BCDY
$(2, 4n, 3)^*$	BCDY
$(3, 3n, 2)^*$	BCDY
$(2, 3, n),\ 6 \leq n \leq 10, n = 15$	PSS, Bruin, Brown, S
$(3, 4, 5)$	S-Stoll
$(5, 5, 7),\ (5, 5, 19),\ (7, 7, 5)$	Dahmen-S

The state of the art : continued

The * here refers to conditional results. For instance, in case $(p, q, r) = (3, 3, n)$, we have no solutions if either $3 \leq n \leq 10^9$, or $n \equiv \pm 2$ modulo 5, or $n \equiv \pm 17$ modulo 78, or

$$n \equiv 51, 103, 105 \text{ modulo } 106,$$

or for n (modulo 1296) one of

$$43, 49, 61, 79, 97, 151, 157, 169, 187, 205, 259, 265, 277, 295,$$
$$313, 367, 373, 385, 403, 421, 475, 481, 493, 511, 529, 583,$$
$$601, 619, 637, 691, 697, 709, 727, 745, 799, 805, 817, 835, 853,$$
$$907, 913, 925, 943, 961, 1015, 1021, 1033, 1051, 1069, 1123,$$
$$1129, 1141, 1159, 1177, 1231, 1237, 1249, 1267, 1285.$$

The state of the art : continued

(p,q,r)	n
$(2,2n,3)$	$3 \leq n \leq 10^7$ or $n \equiv -1 \mod 6$
$(2m,2n,3)$	$m \geq 2$ and $n \equiv -1 \mod 4$
$(2,4n,3)$	$n \equiv \pm 2 \mod 5$ or $n \equiv \pm 2, \pm 4 \mod 13$
$(2,2n,5)$	$n \geq 17$ and $n \equiv 1 \mod 4$ prime
$(3,3n,2)$	$n \equiv 1 \mod 8$ prime.

Methods of proof

These results have primarily followed from either

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We will discuss primarily the latter – the former is a p-adic method for (potentially) determining the rational points on curves of positive genus.

A program for attacking certain $x^p + y^q = z^r$

Given a solution to

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we would like to

1. Construct a "Frey" curve $E_{x,y,z}$ with conductor $N_{x,y,z}$

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2. Connect to this curve a type of modular form of level N
3. Use properties of $E_{x,y,z}$ and the forms at level N to derive arithmetic information

Potential difficulties

1. We are (at present) quite limited in the signatures (p, q, r) for which such a program can be implemented.

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2. Small values of exponents may present problems.
3. We might not derive much (or even any) information!

Possible signatures

Work of Darmon and Granville suggests that restricting attention to Frey-Hellegouarch curves over $\mathbb{Q}$ (or, for that matter, to $\mathbb{Q}$-curves) might enable us to treat only signatures which can be related via descent to one of

$$(p, q, r) \in \{(n, n, n), (n, n, 2), (n, n, 3), (2, 3, n), (3, 3, n)\} .$$

Possible signatures

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$$(p,q,r) \in \{(n,n,n),(n,n,2),(n,n,3),(2,3,n),(3,3,n)\}.$$

There are however, as we shall see, some quite nontrivial examples of ternary equations which may be reduced to the study of the form $Aa^p + Bb^q = Cc^r$ for one of these signatures.

Signature $(n, n, 2)$

Given $Aa^n + Bb^n = Cc^2$, we consider the Frey-Hellegouarch curve

$$E_{a,b,c} \; : \; y^2 = x^3 + 2cCx^2 + BCb^n x,$$

of discriminant $\Delta_E = 64AB^2C^3 \left(ab^2\right)^n$.

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of discriminant $\Delta_E = 64AB^2C^3 \left(ab^2\right)^n$.

Darmon and Merel use this with $A = B = C = 1$ and derive a correspondence between E and an elliptic curve of conductor 32 with complex multiplication.

A new equation via descent

Suppose we have coprime integers a, b and c with

$$a^4 - b^2 = c^n,$$

with $n \geq 7$, say, prime. Then either

$$a^2 - b = r^n \text{ and } a^2 + b = s^n,$$

or

$$a^2 - b = 2^\delta r^n \text{ and } a^2 + b = 2^{n-\delta} s^n,$$

for some integers r and s, and $\delta \in \{1, n-1\}$.

It follows that

$$r^n + s^n = 2a^2 \text{ or } r^n + 2^{n-\delta-1}s^n = a^2,$$

both of which are shown to have no solutions with $|rs| > 1$ in a paper of B-Skinner (for $n \geq 7$). For $n = 5$, the first of these has the solution $(r, s, a) = (3, -1, 11)$.

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The solution $r = s = 1$ to the first equation shows up as a modular form of level 256 (with, again, complex multiplication).

More equations via descent

If, instead, we consider

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Ellenberg uses this approach to show that the above equation has no nontrivial solutions for prime $n \geq 211$ (subsequently reduced to $n \geq 4$ by B-Ellenberg-Ng).

What can go wrong

If we suppose we have a solution to

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then, in general, all we can prove is that a corresponding Frey curve E is congruent modulo n to a particular elliptic curve F of conductor 72.

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This does enable us to conclude that

- $z \equiv 3$ modulo 6, and
- $n > 10^4$, and
- $n \equiv \pm 1$ modulo 5, etc.

The equation $x^3 + y^6 = z^n$

In this case, we can use Frey-Hellegouarch curves to attack both

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In this case, careful examination modulo 7 yields the desired result. From the first Frey-Hellegouarch curve, we are able to show that $7 \mid y$. After some work, we find that the second such curve E necessarily has $a_7(E) = \pm 4$, while $a_7(F) = 0$.

The equation $x^2 + y^4 = z^3$

Coprime integer solutions to this equation necessarily have one of

$$y = \pm(s^2 + 3t^2)\left(s^4 - 18s^2t^2 + 9t^4\right), \text{ or}$$

$$y = 6ts(4s^4 - 3t^4), \text{ or}$$

$$y = 6ts(s^4 - 12t^4), \text{ or}$$

$$y = 3(s-t)(s+t)(s^4 + 8ts^3 + 6t^2s^2 + 8t^3s + t^4),$$

for s and t coprime integers satisfying certain conditions modulo 6.

The equation $a^2 + b^{4n} = c^3$

We may conclude that

$$b^n = 3(s-t)(s+t)(s^4 + 8s^3t + 6s^2t^2 + 8st^3 + t^4),$$

where

$$s \not\equiv t \text{ modulo } 2 \quad \text{and} \quad s \not\equiv t \text{ modulo } 3.$$

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We thus deduce the existence of integers A, B and C for which

$$s - t = A^n, \quad s + t = \frac{1}{3}B^n, \quad s^4 + 8s^3t + 6s^2t^2 + 8st^3 + t^4 = -C^n.$$

It follows that

$$A^{4n} - \frac{1}{27} B^{4n} = 2C^n,$$

with ABC odd and $3 \mid B$. There are (at least) three Frey-Hellegouarch curves we can attach to this Diophantine equation:

$$\begin{aligned} E_1 \, : \, Y^2 &= X(X - A^{4n}) \left(X - \frac{B^{4n}}{27} \right), \\ E_2 \, : \, Y^2 &= X^3 + 2A^{2n} X^2 + 2C^n X, \\ E_3 \, : \, Y^2 &= X^3 - \frac{2B^{2n}}{27} X^2 - \frac{2C^n}{27} X. \end{aligned}$$

The equation $A^{4n} - \frac{1}{27}B^{4n} = 2C^n$

Adding $2B^{4n}$ to both sides of the equation, we find that

$$A^{4n} + \frac{53}{27}B^{4n} = 2(C^n + B^{4n}),$$

and, after some work, that $C + B^4$ is a quadratic non residue modulo 53.

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On the other hand, considering $a_{53}(E_1)$, we find that necessarily

$$(C/B^4)^n \equiv 17 \text{ modulo } 53.$$

This is a contradiction for $n \equiv \pm 2, \pm 4 \mod 13$.

Proposition

(BCDY) If n is a positive integer with

$$n \equiv \pm 2 \text{ modulo } 5 \quad \text{or} \quad n \equiv \pm 2, \pm 4 \text{ modulo } 13,$$

then the equation $a^2 + b^{4n} = c^3$ has only the solution $(a, b, c, n) = (1549034, 33, 15613, 2)$ in positive coprime integers.

A final example : the equation $x^3 + y^{3n} = z^2$

This is a much more subtle case, where we appeal to both parametrizations to $a^3 + b^3 = c^2$ as well as Frey curves attached to $a^2 = b^3 + c^n$

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If, for example, z is odd, the parametrizations imply that

$$b^n = s^4 - 4ts^3 - 6t^2s^2 - 4t^3s + t^4$$

and so

$$b^n = (s-t)^4 - 12(st)^2 = U^4 - 12V^2,$$

to which we attach the $\mathbb{Q}$-curve

$$E_{U,V} : y^2 = x^3 + 2(\sqrt{3}-1)Ux^2 + (2-\sqrt{3})(U^2 - 2\sqrt{3}V)x.$$

The equation $x^3 + y^{3n} = z^2$

After much work, one arrives at . . .

Theorem

If $n \equiv 1 \mod 8$ is prime, then the only solution in nonzero integers to the equation

$$x^3 + y^{3n} = z^2$$

is with $x = 2, y = 1$ and $z = \pm 2$.

Darmon's program

Darmon generalizes the notion of *Frey curve* to that of *Frey abelian variety* to provided a framework for analyzing solutions to

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The technical machinery required to carry out this program for given prime $r > 3$ and arbitrary p is still under development.