On the Diophantine equation $5x^2 + q^{2n} = y^5$

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Abstract: In a joint-paper with A. Laradji and M. Mignotte [1] one of our main results is the following:

Theorem *Let q be an odd prime. If either $q \not\equiv 1 \pmod{600}$ or $q \leq 3 \cdot 10^9$, then there is no integer solution (x, y, n) to the equation*

$$5x^2 + q^{2n} = y^5\,, \quad x, y, n > 0\,. \tag{1}$$

Otherwise, there exists at most one integer solution (x, y, n) and if it actually exists, then it must satisfy the following conditions:
(i) $n < 820$ and $\gcd(n, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13) = 1$.
(ii) There exists an integer v such that $x = 10v(80v^4 - 40v^2 + 1)$, $y = 20v^2 + 1$, $q^n = 2000v^4 - 200v^2 + 1$.

I will mention briefly the tools that were used in the proof; the *modular method* is *not* among them, as all three of us were very far from being masters of this spectacular method. Therefore, the purpose of my lecture is to set the solution of the equation (1) for the remaining unsolved cases as a challenge to the connoisseurs of the modular method. We will be happy if this will end up with the complete solution of equation (1). Even in the case that the modular method does not work for some reason, the profit for the participants of the workshop (mainly the "learners", like me) will hopefully be a deeper insight into the limitations of the method.

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References

[1] A. Laradji, M. Mignotte, N. Tzanakis, On $px^2 + q^{2n} = y^p$ and related Diophantine equations, J. Num. Th. 131 (2011), 1575–1596.